MINI OLYMPIAD

1. Let f be a continuous function on the unit square. Prove that

$$\int_0^1 \left(\int_0^1 f(x,y)dx\right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)dy\right)^2 dx \leq$$

$$\left(\int_0^1 \int_0^1 f(x,y)dxdy\right)^2 + \int_0^1 \int_0^1 f^2(x,y)dxdy.$$

2. Find

$$\min_{a,b\in\mathbb{R}} \max(a^2+b, b^2+a).$$

3. Prove that for all real numbers x,

$$2^x + 3^x - 4^x + 6^x - 9^x \leq 1.$$

4. Find all positive integers n for which the equation $nx^4 + 4x + 3 = 0$ has a real root.

5. If $a_1 + a_2 + ... + a_n = n$ prove that $a_1^4 + a_2^4 + ... + a_n^4 \geq n$.

6. Let $P(x)$ be a polynomial with positive real coefficients. Prove that

$$\sqrt{P(a)P(b)} \geq P(\sqrt{ab}),$$

for all positive real numbers a and b.

7. Show that all real roots of the polynomial $P(x) = x^5 - 10x + 35$ are negative.

8. On a sphere of radius 1 are given four points A, B, C, D such that $AB\cdot AC\cdot AD\cdot BC\cdot BD\cdot CD = \frac{2^9}{3^3}$. Prove that the tetrahedron $ABCD$ is regular.

9. Let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 ... a_n)^{\frac{1}{n}} + (b_1 b_2 ... b_n)^{\frac{1}{n}} \leq ((a_1 + b_1)(a_2 + b_2)...(a_n + b_n))^{\frac{1}{n}}.$$

10. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!n!}{m^m n^n}.$$